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OSCILLATION AND NONOSCILLATION PROPERTIES OF SOLUTIONS OF THIRD ORDER LINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper some necessary and sufficient conditions of third order neutral differential equations are obtained to insure the oscillation of all solutions or converge to zero or tends to infinity as $t \to \infty$. Some examples are given to illustrate the obtained results.

KEYWORDS: Oscillation, Neutral Differential Equations

1. INTRODUCTION

This paper is concerned with the oscillation and nonoscillation criteria of all solution of third order neutral differential equations of the form:

$$\frac{d^3}{dt^3}\Big(y(t) + p(t)y\big(\tau(t)\big)\Big) \quad + q(t)y\big(\sigma(t)\big) = 0 \tag{1.1}$$

Under the following assumptions:

$(\mathbf{A_1})$ $p(t) \in C([t_0,\infty);[0,\infty)), q(t) \in C([t_0,\infty);R),$.

$(\mathbf{A_2})$ $\tau(t), \sigma(t) \in C([t_0,\infty);R), \lim_{t\to\infty}\tau(t) = \infty, \lim_{t\to\infty}\sigma(t) = \infty, \tau(t), \sigma(t)$Are increasing functions

Our aim is to established necessary and sufficient conditions of all proper solutions of equation (1.1) to oscillates or converge to zero or tends to infinity as $t \to \infty$.

By a solution of (1.1), we mean a function $y(t)$ such that $y(t) + p(t)y\big(\tau(t)\big)$ is three times continuously differentiable and $y(t)$ satisfies (1.1) on $[t_0,\infty)$.

A solution is said to be oscillatory if it has arbitrary large zeros, otherwise is said nonoscillatory that is eventually positive or negative. Equation (1.1) is said to be oscillatory if every solution of (1.1) is oscillatory.

There has been a considerable investigation of the oscillation and nonoscillation of third order neutral differential equations. Parhi and Rath (2004) in [11] obtained necessary and sufficient conditions for all bounded solution of

$$\frac{d^n}{dt^n}(y(t) - p(t)y(t-\tau)) \quad + q(t)G(y(t-\sigma)) = 0$$

Oscillate if $\sum_{k=0}^{\infty}\int_{k\tau}^{\infty}(t-k\tau)^{n-1}Q(t)dt = \infty$.Tongxing Li (2010) in [16] obtained sufficient conditions so that the neutral differential equation

$\frac{d^n}{dt^n}[y(t)+p(t)y(\tau(t))] \quad +q(t)G(y(\sigma(t)))=0$

In this paper we obtained sufficient condition for oscillation of all solutions of equation (1.1) or tends to zero or infinity. Some examples are given to illustrate the obtained results.

2. MAIN RESULTS

In this section some results obtained for all solutions of eq. (1.1). For simplicity set

$$z(t)=y(t)+p(t)y(\tau(t)) \tag{2.1}$$

Then eq. (1.1) reduce to

$$z'''(t)=-q(t)y(\sigma(t)) \tag{2.2}$$

Lemma 2.1: ([10], Theorem 2.1.1. pp. 15-16)

Assume that $\tau(t)<t$,and

$\lim\ \inf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds\ >\frac{1}{e}$

Where $p(t),\tau(t)\in C([t_0,\infty);[0,\infty))$ Then

The differential inequality $y'(t)+p(t)y(\tau(t))\le 0$, has no eventually positive solutions.

Theorem 2.1: Suppose that $0\le p(t)\le p<1, \sigma(t)<t<\tau(t), q(t)\ge 0$, and

$$\lim\ \sup_{t\to\infty}\frac{1}{2}\int_{\sigma(t)}^{t}\left(v-\sigma(t)\right)^2 q(v)(1-p(\sigma(v)))dv>1\ ) \tag{2.3}$$

Then every solution $y(t)$of equation (1.1) is either oscillatory or$|y(t)|$ tends to infinity as$t\to\infty$.

Proof: Assume that $y(t)$ be an eventually positive solution of (1.1) (When $y(t)$ is eventually negativethe proof is similar and will be omitted)

From eq. (2.2), we get$z'''(t)\le 0\ , t\ge t_0$, hence

$z''(t), z'(t), z(t)$ Are monotone functions

We claim that $z''(t)>0$ otherwise $z'(t)<0, z(t)<0$, which is impossible, since$z(t)>0$, we have two cases for $z'(t)$:

Case1: $z'(t)>0, z(t)>0, t\ge t_1\ge t_0$ and $\lim_{t\to\infty}z(t)=\infty$

$z(t)=y(t)+p(t)y(\tau(t))\le y(t)+py(\tau(t))$, which implies that$\lim_{t\to\infty}y(t)=\infty$,

Otherwise $y(t)\le k, k>0, z(t)\le k+pk=k(1+p)<\infty$.

Case 2: $z'(t)<0, z(t)>0, t\ge t_1\ge t_0$

From (2.1), we get$y(t)=z(t)-p(t)z(\tau(t))+p(t)p(\tau(t))y(\tau(\tau(t)))$

$$y(\sigma(t))\ge\left(1-p(\sigma(v))\right)z(\sigma(t)) \tag{2.4}$$

Consider the integral equality

$$z^{(k)}(s) = \sum_{i=k}^{n-1}(-1)^{i-k}\frac{(t-s)^{i-k}}{(i-k)!}z^{(i)}(t) + \frac{(-1)^{n-k}}{(n-1-k)!}\int_s^t (v-s)^{n-1-k}z^{(n)}(v)dv \qquad (2.5)$$

Where $s \le v \le t$ and $k \in \{0,1,2\}$, let $k = 0, n = 3$, we get from (2.5)

$$z(s) - z(t) = -(t-s)z'(t) + \frac{(t-s)^2}{2}z''(t) - \frac{1}{2}\int_s^t (v-s)^2 z'''(v)dv$$

$$\ge -\frac{1}{2}\int_s^t (v-s)^2 z'''(v)dv = \frac{1}{2}\int_s^t (v-s)^2 q(v)y(\sigma(v))dv$$

$$\ge \frac{1}{2}\int_s^t (v-s)^2 q(v)\left(1 - p(\sigma(v))\right)z(\sigma(v))dv$$

$$\ge \frac{z(\sigma(t))}{2}\int_s^t (v-s)^2 q(v)\left(1 - p(\sigma(v))\right)dv$$

$$z(s) \ge \frac{z(\sigma(t))}{2}\int_s^t (v-s)^2 q(v)\left(1 - p(\sigma(v))\right)dv$$

Let $s = \sigma(t)$, then the last inequality leads to

$$1 \ge \frac{1}{2}\int_{\sigma(t)}^t \left(v - \sigma(t)\right)^2 q(v)\left(1 - p(\sigma(v))\right)dv,$$

Which contradicts (2.3)

Remark 2.1: I fall conditions of Theorem 2.1 hold except the condition (2.3) replaced by

$$\liminf_{t\to\infty}\int_{\sigma(t)}^t \int_s^{\alpha(s)} \frac{(v-s)\, q(v)}{p\left(\tau^{-1}(\sigma(v))\right)}dv\, ds > \frac{1}{e} \qquad (2.6)$$

Then the conclusion of Theorem 2.1 remains true.

Proof: The proof is similar to the proof of Theorem 2.1up to case 2, it remain to proof case 2.

Case 2: $z(t) > 0,\ z'(t) < 0, z''(t) > 0, z'''(t) \le 0, t \ge t_1 \ge t_0$.

From eq. (2.1) it follows that (2.4) holds that is

$$y(\sigma(t)) \ge \left(1 - p(\sigma(v))\right)z(\sigma(t))$$

From (2.5), let $k = 1$ we get

$$z'(s) = z'(t) - (t-s)z''(t) + \int_s^t (v-s)z'''(v)dv, t > s$$

$$\le -\int_s^t (v-s)q(v)y(\sigma(v))dv$$

$$\le -\int_s^t (v-s)q(v)(1 - p(\sigma(v)))z(\sigma(v))dv$$

$$\le -z(\sigma(t))\int_s^t (v-s)q(v)(1 - p(\sigma(v)))dv$$

Let $s = \alpha(t) < t$

$$z'(s) \le -z(\sigma(t))\int_{\alpha(t)}^t (v-s)(1 - p(\sigma(v)))q(v)dv$$

$z'(s) + z(\sigma(t)) \int_{\alpha(t)}^{t} (v-s)(1-p(\sigma(v)))q\,(v)dv \le 0$

Then by Lemma 2.1.2, with virtue of condition (2.6) the last inequality cannot have eventually positive solution which is a contradiction.

Example 1: Consider the third order neutral differential equation:

$$\frac{d^3}{dt^3}\left(y(t) + (\tfrac{1}{2}+\tfrac{1}{3}\cos 4t)\, y(t+2\pi)\right) + \left(\tfrac{3}{2}+\tfrac{1}{3}\cos 4t\right) y\left(t-\tfrac{3\pi}{2}\right) = 0, t \ge t_0 (E1)$$

Where $p(t) = \frac{1}{2}+\frac{1}{3}\cos 4t, \tau(t) = t+2\pi$, $\sigma(t) = t - \frac{3\pi}{2}$ and $q(t) = \frac{3}{2}+\frac{1}{3}\cos 4t$.

One can find that all conditions of Theorem 2.1 hold.

To see condition (2.4):

$$\limsup_{t\to\infty} \frac{1}{2}\int_{\sigma(t)}^{t} (v-\sigma(t))^2 q(v)\left(1-p(\sigma(v))\right) dv =$$

$$= \lim_{t\to\infty} \frac{1}{2}\int_{t-\frac{3\pi}{2}}^{t} \left(v-t+\frac{3\pi}{2}\right)^2 \left(\frac{3}{2}+\frac{1}{3}\cos 4v\right)\left(\frac{1}{2}-\frac{1}{3}\cos 4v\right) dv = 3.39 > 1$$

By Theorem 2.1 every solution of eq.(E1) either oscillatory or $|y(t)| \to \infty$ as $t \to \infty$, for instance $y(t) = \frac{\cos t}{\frac{3}{2}+\frac{1}{2}\cos 4t}$ is such a solution.

Theorem 2.2: Suppose that $0 \le p(t) \le p < 1, \sigma(t) > t > \tau(t), q(t) \le 0$, and

$$\liminf_{t\to\infty} \int_{t}^{\sigma(t)} \int_{s}^{\alpha(s)} (v-s)|q(v)|(1-p(\sigma(v)))\, dv ds > \frac{1}{e} \tag{2.7}$$

Where $\alpha(t) \in C(R,R), \alpha(t) > t$

Then every solution of equation (1.1) is either oscillatory or nonoscillatory tends to zero or tends to infinity as $t \to \infty$.

Proof: Assume that $y(t)$ be an eventually positive solution of (1.1) (the proof when $y(t)$ is eventually negative is similar and will be omitted)

From eq. (2.2), we get $z'''(t) \ge 0$, $t \ge t_0$, hence

$z''(t), z'(t), z(t)$, are monotone functions

We have two cases for $z''(t)$:

Case 1: $z''(t) > 0, t \ge t_1 \ge t_{0,}$

Hence $z'(t) > 0, z(t) > 0$ and $\lim_{t\to\infty} z(t) = \infty$, implies that $\lim_{t\to\infty} y(t) = \infty$, otherwise $y(t) \le k, k > 0$, then

$z(t) = y(t) + p(t)y(\tau(t)) \le k + pk \le (1+p)k < \infty$.

Case 2: $\mathbf{z}''(t) < 0, t \ge t_1 \ge t_{0,}$

We claim that $z'(t) > 0$, otherwise $z'(t) < 0$, hence $z(t) < 0$, which is a contradiction

So our claim has been established and

$z'(t) > 0, z(t) > 0,$

From (2.1) it follows that (2.4) holds that is $y(\sigma(t)) \geq \left(1 - p(\sigma(t))\right) z(\sigma(t))$

If k =1, in (2.5) we get

$z'(t) = z'(s) - (s-t)z''(s) + \int_t^s (v-t)z'''(v)dv,$

$z'(t) \geq \int_t^s (v-t)|q(v)|y(\sigma(v))\, dv$

$\geq \int_t^s (v-t)|q(v)|(1 - p(\sigma(v)))z(\sigma(v))\, dv$

$\geq z(\sigma(t)) \int_t^s (v-t)|q(v)|(1 - p(\sigma(v)))\, dv$

Let $s = \alpha(t) > t$, then the last inequality leads to

$z'(t) \geq z(\sigma(t)) \int_t^{\alpha(t)} (v-t)|q(v)|(1 - p(\sigma(v)))\, dv$

$z'(t) - z(\sigma(t)) \int_t^{\alpha(t)} (v-t)|q(v)|(1 - p(\sigma(v)))\, dv \geq 0$

Then by lemma 2.1.2-ii...with virtue of condition (2.7) the last inequality cannot have eventually positive solution which is a contradiction.

Remark 2.2: We can replace the condition (2.7) in theorem 2.2 by the condition

$$\limsup_{t\to\infty} \frac{1}{2}\int_{\sigma^{-1}(t)}^{t} \left(v - \sigma^{-1}(t)\right)^2 |q(v)|(1 - p(\sigma(v)))dv > 1 \quad (2.8)$$

And the conclusion of theorem remains true.

Proof: The proof is similar to the proof of theorem 2.2.up case 2, to proof this case.

From eq. (2.1), we get$z'''(t) = -q(t)y(\sigma(t)) \geq 0\, , t \geq t_0$, hence$z''(t), z'(t), z(t)$, are monotone functions, we have only the following case to consider:

$z(t) > 0, z'(t) > 0, z''(t) < 0, t \geq t_1 \geq t_{0,},$

From eq. (2.2) it follows (2.4) holds that is $y(\sigma(t)) \geq (1 - p(\sigma(t))z(\sigma(t))$

From (2.5), when$k = 0$, we get

$z(s) - z(t) = -(t-s)z'(t) + \frac{(t-s)^2}{2} z''(t) - \frac{1}{2}\int_s^t (v-s)^2 z'''(v)dv$

$-z(t) \leq \frac{-1}{2}\int_s^t (v-s)^2|q(v)|y(\sigma(v))dv$

$z(t) \geq \frac{1}{2}\int_s^t (v-s)^2|q(v)|y(\sigma(v))dv$

Let $s = \sigma^{-1}(t) < t$, then the last inequality will be

$z(t) \geq \frac{1}{2}\int_{\sigma^{-1}(t)}^{t} (v - \sigma^{-1}(t))^2|q(v)|(1 - p(\sigma(v))z(\sigma(v))dv$

$$z(t) \geq \frac{z(t)}{2}\int_{\sigma^{-1}(t)}^{t}(v-\sigma^{-1}(t))^2|q(v)|(1-p(\sigma(v))dv$$

$$1 \geq \frac{1}{2}\int_{\sigma^{-1}(t)}^{t}\left(v-\sigma^{-1}(t)\right)^2|q(v)|(1-p(\sigma(v))dv,$$

Which contradicts (2.8)

Example 2: Consider the third order neutral differential equation:

$$\frac{d^3}{dt^3}\left(y(t)+(\frac{1}{3}+\frac{1}{3}\sin 4t)\,y(t-2\pi)\right)-\left(\frac{4}{3}+\frac{1}{3}\sin 4t\right)y\left(t+\frac{3\pi}{2}\right)=0, t \geq t_0, (E2)$$

Where$p(t)=\frac{1}{3}+\frac{1}{3}\sin 4t$,$\tau(t)=t-2\pi$, $\sigma(t)=t+\frac{3\pi}{2}$ and $q(t)=-\left(\frac{4}{3}+\frac{1}{3}\sin 4t\right)$.

One can find that all conditions of Theorem 2.2 are holds.

To see condition (2.7): Let $\alpha(s)=s+1>s$

$$\liminf_{t\to\infty}\int_t^{\sigma(t)}\int_s^{\alpha(s)}(v-s)|q(v)|(1-p(\sigma(v)))\,dvds=$$

$$=\lim_{t\to\infty}\int_t^{t+\frac{3\pi}{2}}\int_s^{s+1}(v-s)\,(\frac{4}{3}+\frac{1}{3}\sin 4v)\,(\frac{2}{3}-\frac{1}{3}\sin 4v)\,dvds$$

$$=\lim_{t\to\infty}\int_t^{t+\frac{3\pi}{2}}\frac{1}{6}\,ds=\frac{3\pi}{12}=0.785>\frac{1}{e}$$

By Theorem 2.2 every solution of eq.(E2) either oscillatory or $y(t)$ tends to infinity as $t\to\infty$, for instance$y(t)=\frac{\sin t}{\frac{4}{3}+\frac{1}{3}\sin 4t}$is such a solution.

REFERENCES

1. D. D. Bainov and D. P. Mishev: Oscillation Theory for Neutral Differential Equations with Delay. Adam Hilger, Bristol, 1991.
2. T. Candan and R. S. Dahiya: Oscillation Properties of Nonlinear Neutral Differential Equations of n-th Order. International Journal of Mathematics and Mathematical Science, 71(2004), 3893-3900.
3. P. Das, B. B. Mishra and C. R. Dash: Oscillation Theorems for Neutral Delay Differential Equations of Odd Order. Bulletin of the Institute of Mathematics.Vol.1, No. 4 (2006), 557-568.
4. J. G. Dix: Oscillation of solutions to a neutral differential equation involving an n-order operator with variable coefficients and a forcing term... Differential Equations Dynamical Systems, 22 (2014), 15-31.
5. I. Gyori and G. Ladas: Oscillation Theory of Delay Differential Equations. *Clarendon Press, Oxford,* 1991.
6. J. Jaros and T. Kusano: On Oscillation of Linear Neutral Differential Equations of Higher Order. *Hiroshima Math. J.* 20 (1990), 407- 419.
7. J.Jaros and T.Kusano: On a class of first order nonlinear functional differential equation of neutral type, Czechoslovak Mathematical Journal, Vol. 40, No.3,(1990), 475-490
8. B. Karpuz,Ö. Öcalan and S. Özturk: Comparison theorems on the oscillation and asymptotic behaviour of higher-order neutral differential equations. *Glasgow Math. Journal*, 52 (2010), 107-114.

9. G. Ladas and I. P. Stavroulankis: On Delay Differential Inequalities of Higher Order. Canad. Math. Bull. Vol. 25, No. 3, (1982), 348-354.

10. G. S. Ladde, V. Lakshmikanthamand B.G. Zhang: Oscillation Theory of Differential Equations with Deviating Arguments. Marcel Dekker, Inc, 1987.

11. T. Li, Z. Han, P. Zhao, and S. Sun: Oscillation of Even-Order Neutral Delay Differential Equations Advances in Difference Equations, (2010), 1- 9.

12. H. A. Mohammad: Oscillation of Linear Neutral Differential Equation of Third Order, Iraqi Journal of Science, Vol.50, No.4, 2009, 543-547.

13. N. Parhi, R. N. Rath: Oscillation Behaviour of Solutions of Nonlinear Higher Order Neutral Differential Equations. Mathematica Bohemica. Vol. 129, No. 1,(2004),11-27.

14. R.N. Rath, L.N. Padhy and N. Misra; Oscillation of solution of nonlinear neutral delay differential equations of higher order for$p(t) = \pm 1$. ArchivumMathematicumVol. 40, (2004), 359-366.

15. M.K. Yildiz: Oscillation results of higher order nonlinear neutral delay differential equations with oscillating coefficients. Advance in Dynamical system and application, Vol. 3, No. 2, (2008), 297-303.

16. A. Zafer: Oscillation Criteria for Even Order Neutral Differential Equations. *Appl. Math. Lett.* Vol. 11, No. 3, (1998), 21-25.